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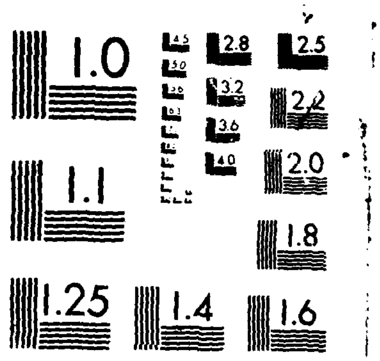

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Consider the model of multiple superimposed exponential signals in additive Gaussian noise

$$y_j(t) = \sum_{i=1}^{p} s_{ij}\lambda_i^t + e_j(t), \quad t = 0,1,\ldots,n-1, \quad j = 1,\ldots,N$$

20 (continued)

where N is fixed and $n \to \infty$, $\lambda_i = \exp(\sqrt{-1}\omega_i)$, $\omega_i \in [0,2\pi)$, $i = 1,\ldots,p$, ω_i, s_{ij} are unknown parameters and p is known. Further, $e_j(t) = e_{j1}(t) + \sqrt{-1}e_{j2}(t)$, and $e_{j1}(t)$, $e_{j2}(t)$, $t = 0,1,2,\ldots$, $j = 1,\ldots,N$, are mutually independent and identically distributed real random variables with a common distribution $N(0,\sigma^2/2)$, $0 < \sigma^2 < \infty$, σ^2 is unknown. It is shown that if $\omega_i \neq \omega_j$ when $i \neq j$ and $\sum_{j=1}^{N}|s_{ij}| > 0$ for $i = 1,\ldots,p$, then the Maximum Likelihood estimate $(\hat{\lambda}_1, \ldots, \hat{\lambda}_p)$ is strongly consistent. Moreover, it is shown that $\hat{\lambda}_i$ converges to λ_i with an exponential rate.

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ABSTRACT

Consider the model of multiple superimposed exponential signals in additive Gaussian noise

$$Y_j(t) = \sum_{i=1}^{p} s_{ij}\lambda_i^t + e_j(t), \quad t = 0,1,\ldots,n-1, \quad j = 1,\ldots,N,$$

where N is fixed and $n \to \infty$, $\lambda_i = \exp(\sqrt{-1}\omega_i)$, $\omega_i \in [0,2\pi)$, $i = 1,\ldots,p$, ω_i, s_{ij} are unknown parameters and p is known. Further, $e_j(t) = e_{j1}(t) + \sqrt{-1}e_{j2}(t)$, and $e_{j1}(t)$, $e_{j2}(t)$, $t = 0,1,2,\ldots$, $j = 1,\ldots,N$, are mutually independent and identically distributed real random variables with a common distribution $N(0,\sigma^2/2)$, $0 < \sigma^2 < \infty$, σ^2 is unknown. It is shown that if $\omega_i \neq \omega_j$ when $i \neq j$ and $\sum_{j=1}^{N}|s_{ij}| > 0$ for $i = 1,\ldots,p$, then the Maximum Likelihood estimate $(\hat{\lambda}_1, \ldots, \hat{\lambda}_p)$ is strongly consistent. Moreover, it is shown that $\hat{\lambda}_i$ converges to λ_i with an exponential rate.

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Key words and phrases: consistency, exponential rate, Maximum Likelihood estimate, signal processing.

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1. INTRODUCTION

Consider the following model of multiple superimposed exponential signals in additive Gaussian noise

$$Y_j(t) = \sum_{i=1}^{p} s_{ij}\lambda_i^t + e_j(t), \quad t = 0,1,\ldots,n-1, \quad j = 1,\ldots,N \tag{1}$$

where $\lambda_i = \exp(\sqrt{-1}\omega_i)$, $i = 1,\ldots,p$, $\omega_i \in [0,2\pi)$, ω_i, s_{ij} are unknowns and p is assumed to be known. Further, $e_j(t) = e_{j1}(t) + \sqrt{-1}e_{j2}(t)$ and $e_{j1}(t)$, $e_{j2}(t)$, $t = 0,1.2,\ldots$, $j = 1,\ldots,N$, are mutually independent and identically distributed (iid.) real random variables with a common distribution $N(0,\sigma^2/2)$, $0 < \sigma^2 < \infty$, σ^2 is unknown.

Quite a number of papers appeared dealing with the estimation of parameters in this model, which is important in problems related to signal processing and time series analysis. When $\lambda_1, \ldots, \lambda_p$ are known, (1) reduces to an ordinary linear regression model in which s_{ij}'s are usually estimated by the Least Squares method. Therefore a conceivable way to handle the estimation problem in (1) is as follows: Obtain by some way an estimator $(\hat{\lambda}_1, \ldots, \hat{\lambda}_p)$ of $(\lambda_1, \ldots, \lambda_p)$. Substitute $\hat{\lambda}_i$ for λ_i in (1), consider the $\hat{\lambda}_i$'s as known constants and use the LS method to yield an estimate for s_{ij}. This seemingly reasonable procedure has the drawback that the estimate of s_{ij} thus obtained is usually non-consistent, as indicated in [1].

For the more important problem of estimating $\lambda_1, \ldots, \lambda_p$, several methods have been proposed in the literature. Bresler and Macovski derived in [2] the LS criterion in the form of minimizing some function not involving s_{ij}. Under the normality assumption here, it is the same as the Maximum Likelihood criterion. Their method consists in introducing a polynomial $b_0 + b_1 z + \ldots + b_p z^p$ having $\lambda_1, \ldots, \lambda_p$ as roots, thus reducing the problem of estimating $\lambda_1, \ldots, \lambda_p$ to that of estimating the coefficient vector $\underset{\sim}{b} = (b_0, \ldots, b_p)'$. Specifically, define the set

$$B = \left\{ \underset{\sim}{b} : \sum_{i=1}^{p} |b_i|^2 = 1, \quad b_p \geq 0 \right\} \tag{2}$$

and the $(n-p)\times n$ matrix

$$B_n(\underset{\sim}{b}) = \begin{pmatrix} b_0 & b_1 & \cdot & \cdot & \cdot & b_p & & & 0 \\ & b_0 & b_1 & \cdot & \cdot & \cdot & b_p & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & & & b_0 & b_1 & \cdot & \cdot & \cdot & b_p \end{pmatrix}, \tag{3}$$

$$D_n(\underset{\sim}{b}) = B_n(\underset{\sim}{b}) B_n^*(\underset{\sim}{b}),$$

where $B_n^*(\underset{\sim}{b})$ denotes the conjugate transpose of $B_n(\underset{\sim}{b})$. Also,

$$\underset{\sim}{Y}(j,n) = \Big(Y_j(0),\ Y_j(1),\ \ldots,\ Y_j(n-1)\Big)', \quad j = 1, \ldots, N$$

$$Q_n(Y,\underset{\sim}{b}) = \sum_{j=1}^{N} \underset{\sim}{Y}^*(j,n) B_n^*(\underset{\sim}{b}) D_n^{-1}(\underset{\sim}{b}) B_n(\underset{\sim}{b}) \underset{\sim}{Y}(j,n) \tag{4}$$

Bresler and Macovski showed in [2] that the vector $\hat{\underset{\sim}{b}} = \hat{\underset{\sim}{b}}(n)$ minimizing Q_n on B, that is to say,

$$Q_n(Y,\hat{\underset{\sim}{b}}) = \min_{\underset{\sim}{b} \in B} Q_n(Y,\underset{\sim}{b}) \tag{5}$$

is the ML estimate of $\underset{\sim}{b}^{(0)} = (b_0^{(0)}, \ldots, b_p^{(0)})' \in B$, where

$$b_0^{(0)} + b_1^{(0)} z + \ldots + b_p^{(0)} z^p = 0, \tag{6}$$

has roots $\lambda_1, \ldots, \lambda_p$. Bresler and Macovski suggested an iterative process to compute $\hat{\underset{\sim}{b}}$. No proof is given for the convergence of this process. Nor is one guaranteed that when the process does converge, the limit is indeed an overall minimizing point of Q_n, and not a local minimum.

Theoretically it is interesting to give a close study of the statistical properties of the ML estimate $\hat{\underset{\sim}{b}}$. For, although the ML method usually gives statistical procedures with satisfactory performance, in particular when the normality assumption is in force as here, the complexity of the model (from the point of view that the unknowns of the model appear in rather complicated expressions) makes it unclear how good the ML estimate is under the present situation. As mentioned earlier, under model (1) the ML estimate of s_{ij} is not even consistent. So also the good performance of the ML estimate of $(\lambda_1, \ldots, \lambda_p)$ cannot be taken for granted.

This paper is devoted to a basic problem of the asymptotic theory of the ML estimate of $(\lambda_1, \ldots, \lambda_p)$ — its consistency. On reducing the problem to the estimation of $\underset{\sim}{b}^{(0)}$ as described earlier, it is seen that the problem is equivalent to the consistency of the ML estimate $\hat{\underset{\sim}{b}} = \hat{\underset{\sim}{b}}(n)$ of $\underset{\sim}{b}^{(0)}$. Our main result is the following theorem:

THEOREM 1. Suppose the following conditions are satisfied:

1. $|\lambda_1| = \ldots = |\lambda_p| = 1$, $\lambda_i \neq \lambda_j$ for $i \neq j$.
2. For each $k = 1,\ldots,p$, $\sum_{j=1}^{N} |s_{kj}| > 0$.
3. $\{e_j(t)\}$ satisfies the conditions elaborated at the beginning of this section.

Then for arbitrarily given $\varepsilon > 0$, there exists constant $c > 0$ independent of n such that

$$P(\|\hat{\underset{\sim}{b}}(n) - \underset{\sim}{b}^{(0)}\| \geq \varepsilon) \leq e^{-cn} \tag{7}$$

for n large, where $\|\underset{\sim}{a}\|$ denotes the Euclidean length of the vector $\underset{\sim}{a}$.

(7) entails, in view of the well-known Borel-Cantelli lemma, that $\hat{\underset{\sim}{b}} = \hat{\underset{\sim}{b}}(n)$ is a strongly consistent estimate of $\underset{\sim}{b}^{(0)}$.

2. LEMMAS

Some facts concerning mainly with the matrix $D_n(\underset{\sim}{b})$ will be needed in proving the theorem. For convenience we shall write

$$m = n - p. \tag{8}$$

LEMMA 1. For any $\underset{\sim}{b} \in B$, we have

$$D_n^{-1}(\underset{\sim}{b}) \geq \frac{1}{p+1} I_m, \tag{9}$$

$$D_n^{-1}(\underset{\sim}{b}) \leq 2^{-p(p+1)}(p+1)^p n^{3(p+1)^2} I_m, \tag{10}$$

where I_m is the identity matrix of order m.

Proof. (9) follows from $\operatorname{tr}\left(B_n(\underset{\sim}{b})B_n^*(\underset{\sim}{b})\right) = p + 1$.

To prove (10), we proceed to find the minimum

$$H \equiv \min_{\underset{\sim}{b} \in B} \min_{\underset{\sim}{u} \in A} \underset{\sim}{u}^* D_n(\underset{\sim}{b})\underset{\sim}{u} \tag{11}$$

where A is the set $\{\underset{\sim}{u} = (u_0, \ldots, u_{m-1})' : \sum_{i=0}^{m-1} |u_i|^2 = 1\}$.

Introduce the $(p+1)\times n$ matrix $U^*(\underset{\sim}{u})$:

$$U^*(\underset{\sim}{u}) = \begin{pmatrix} \bar{u}_0 & \bar{u}_1 & \cdot & \cdot & \cdot & \bar{u}_{m-1} & \cdot & & 0 \\ & \bar{u}_0 & \bar{u}_1 & \cdot & \cdot & \cdot & \bar{u}_{m-1} & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & & & \bar{u}_0 & \bar{u}_1 & \cdot\cdot & \cdot & \cdot & \bar{u}_{m-1} \end{pmatrix}$$

One sees easily that

$$\underset{\sim}{u}^* B_n(\underset{\sim}{b}) = \underset{\sim}{b}' U^*(\underset{\sim}{u}).$$

Hence,

$$\underset{\sim}{u}^* D_n(\underset{\sim}{b})\underset{\sim}{u} = \underset{\sim}{b}' V(\underset{\sim}{u})\overline{\underset{\sim}{b}}, \qquad V(\underset{\sim}{u}) = U^*(\underset{\sim}{u})U(\underset{\sim}{u})$$

and

$$H = \min_{\underset{\sim}{u}\in A} \min_{\underset{\sim}{b}\in B} \underset{\sim}{b}' V(\underset{\sim}{u})\overline{\underset{\sim}{b}}. \tag{12}$$

Now we prove that for any $\underset{\sim}{u} \in A$, we have

$$\det V(\underset{\sim}{u}) \geq 2^{p(p+1)} n^{-3(p+1)^2}. \tag{13}$$

Since (13) is true when $p = 0$, suppose that $p \geq 1$ and $n \geq p + 1 \geq 2$. Define

$$f_n(x) = \overline{u}_0 + \overline{u}_1 x + \dots + \overline{u}_{m-1} x^{m-1}$$

and $\omega = \exp(\sqrt{-1}\, 2\pi/n)$, $\Omega = \{\omega, \omega^2, \dots, \omega^n\}$. There exist at least $p + 1$ elements $\omega_1, \dots, \omega_{p+1}$ in Ω, such that

$$|f_n(\omega_k)| \geq n^{-(p+1)}, \quad k = 1,2,\dots,p+1. \tag{14}$$

Indeed, supposing in the contrary that

$$\phi_k \in \Omega, \quad |f_n(\phi_k)| < n^{-(p+1)}, \quad k = 1,\dots,n-p\ (=m), \tag{15}$$

then, on putting

$$\Omega - \{\phi_1, \dots, \phi_m\} = \{\phi_{m+1}, \dots, \phi_n\},$$

$$\Lambda = \{1, 2, \dots, m\}, \qquad \Lambda_1 = \{m+1, \dots, n\},$$

and using Lagrange interpolation formula, we have

$$f_n(e^{\sqrt{-1}\,\theta}) = \sum_{j=1}^{m} f_n(\phi_j) \prod_{k\in\Lambda-\{j\}} (e^{\sqrt{-1}\theta} - \phi_k) \Big/ \prod_{k\in\Lambda-\{j\}} (\phi_j - \phi_k). \tag{16}$$

For fixed $j \in \Lambda$, we have

$$\prod_{k\in\Lambda-\{j\}}(\phi_j - \phi_k)\prod_{k\in\Lambda_1}(\phi_j - \phi_k) = \lim_{x\to\phi_j}\frac{x^n - 1}{x - \phi_j} = n\phi_j^{n-1}.$$

Hence,

$$\left|\prod_{k\in\Lambda-\{j\}}(\phi_j - \phi_k)\right|^{-1} \leq \frac{1}{n}\left|\prod_{k\in\Lambda_1}(\phi_j - \phi_k)\right| \leq 2^p/n. \tag{17}$$

Two cases are possible: First,

$$|\theta - \arg\phi_k| \geq \pi/n \pmod{2\pi}, \quad k \in \Lambda_1 U\{j\}.$$

In this case we have

$$\left|\prod_{k\in\Lambda_1 U\{j\}}(e^{\sqrt{-1}\,\theta} - \phi_k)\right| \geq \left|\sin\frac{\pi}{n}\right|^{p+1}.$$

Since $\sin\frac{\pi}{n} \geq \frac{2}{\pi}\frac{\pi}{n} = \frac{2}{n}$ when $n \geq 2$, we have

$$\left|\prod_{k\in\Lambda-\{j\}}(e^{\sqrt{-1}\,\theta} - \phi_k)\right| = \left|e^{\sqrt{-1}\,n\theta} - 1\right|\left|\prod_{k\in\Lambda_1 U\{j\}}(e^{\sqrt{-1}\,\theta} - \phi_k)\right|^{-1}$$

$$\leq 2\left|\sin\frac{\pi}{n}\right|^{-p-1} \leq 2^{-p}n^{p+1}. \tag{18}$$

Second,

$$|\theta - \arg\phi_\ell| < \pi/n \pmod{2\pi}, \quad \text{for some} \quad \ell \in \Lambda_1 U\{j\}.$$

(Note that there are at most one such ℓ.) In this case, noticing that $|\theta - \arg\phi_k| \geq \pi/n \pmod{2\pi}$ for any $k \in \Lambda_1 U\{j\}$, $k \neq \ell$, and that $|e^{\sqrt{-1}\,n\theta} - 1|/|e^{\sqrt{-1}\,\theta} - \phi_\ell| \leq n$, we have

$$\left|\prod_{k\in\Lambda-\{j\}}(e^{\sqrt{-1}\,\theta}-\phi_k)\right| = \left|e^{\sqrt{-1}\,n\theta}-1\right|\left|\prod_{k\in\Lambda_1\cup\{j\}}(e^{\sqrt{-1}\,\theta}-\phi_k)\right|^{-1}$$

$$\leq n\left|\prod_{k\in\Lambda_1\cup\{j\},\ k\neq\ell}(e^{\sqrt{-1}\,\theta}-\phi_k)\right|^{-1}$$

$$\leq n\left(\sin\frac{\pi}{n}\right)^{-p} \leq 2^{-p}n^{p+1}. \tag{19}$$

From (15)-(19), we obtain

$$\left|f_n(e^{\sqrt{-1}\,\theta})\right| < nn^{-(p+1)}(2^p n^{-1})2^{-p}n^{p+1} = 1, \quad \text{for all} \quad \theta \in [-\pi,\pi).$$

Therefore,

$$\sum_{j=0}^{m-1}|u_j|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f_n(e^{\sqrt{-1}\,\theta})|^2 d\theta < 1,$$

contradicting the fact that $\sum_{j=0}^{m-1}|u_j|^2 = 1$. This proves (14).

Now put (remember that $\omega = e^{\sqrt{-1}\,2\pi/n}$)

$$\underset{(n\times n)}{G} = \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \omega & \omega^2 & \cdot & \cdot & \cdot & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdot & \cdot & \cdot & \omega^{2(n-1)} \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdot & \cdot & \cdot & \omega^{(n-1)^2} \end{pmatrix},$$

$$\underset{((p+1)\times n)}{F} = \begin{pmatrix} f_n(1) & f_n(\omega) & \cdots & f_n(\omega^{n-1}) \\ f_n(1) & \omega f_n(\omega) & \cdots & \omega^{n-1} f_n(\omega^{n-1}) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_n(1) & \omega^p f_n(\omega) & \cdots & \omega^{p(n-1)} f_n(\omega^{n-1}) \end{pmatrix},$$

$$\underset{((p+1)\times(p+1))}{F_1} = \begin{pmatrix} f_n(\omega_1) & f_n(\omega_2) & \cdots & f_n(\omega_{p+1}) \\ \omega_1 f_n(\omega_1) & \omega_2 f_n(\omega_2) & \cdots & \omega_{p+1} f_n(\omega_{p+1}) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \omega_1^p f_n(\omega_1) & \omega_2^p f_n(\omega_2) & \cdots & \omega_{p+1}^p f_n(\omega_{p+1}) \end{pmatrix}.$$

Then

$$U^*(\underset{\sim}{u})U(\underset{\sim}{u}) = \frac{1}{n} U^*(\underset{\sim}{u})GG^*U(\underset{\sim}{u}) = \frac{1}{n} FF^* \geq \frac{1}{n} F_1F_1^*.$$

Hence,

$$\begin{aligned} \det V(\underset{\sim}{u}) &= \det\left(U^*(\underset{\sim}{u})U(\underset{\sim}{u})\right) \geq n^{-(p+1)} |\det F_1|^2 \\ &= n^{-(p+1)} \prod_{k=1}^{p+1} \left|f_n(\omega_k)\right|^2 \prod_{1\leq \ell<q\leq p+1} \left|\lambda_\ell - \omega_q\right|^2 \\ &\geq n^{-(p+1)} n^{-2(p+1)^2} \left|\sin \frac{\pi}{n}\right|^{p(p+1)} \\ &\geq n^{-(p+1)} n^{-2(p+1)^2} \left(\frac{2}{n}\right)^{p(p+1)} = 2^{p(p+1)} n^{-3(p+1)^2} \end{aligned}$$

and (13) is proved.

Denote by $L(\underset{\sim}{u})$ and $\ell(\underset{\sim}{u})$ the largest and smallest eigenvalue of $V(\underset{\sim}{u})$, respectively. Obviously we have $L(\underset{\sim}{u}) \leq p + 1$, for any $\underset{\sim}{u} \in A$. From this and (13), we have

$$\ell(\underset{\sim}{u}) \geq C_p n^{-3(p+1)^2}, \qquad C_p = 2^{p(p+1)}(p+1)^{-p}.$$

From this and (12), we obtain

$$H \geq C_p n^{-3(p+1)^2}. \tag{20}$$

Now denote by $d_n(\underset{\sim}{b})$ the smallest eigenvalue of $D_n(\underset{\sim}{b})$. By (11), (20), we see that $d_n(\underset{\sim}{b}) \geq C_p n^{-3(p+1)^2}$, for any $\underset{\sim}{b} \in B$, which amounts to the same thing as (10). Lemma 1 is proved.

LEMMA 2. For arbitrarily given $h > 0$, there exists $h_1 > 0$ such that for any $\underset{\sim}{b} \in B$, $\tilde{\underset{\sim}{b}} \in B$ with $|\underset{\sim}{b} - \tilde{\underset{\sim}{b}}| \leq n^{-h_1}$, we have

$$|D_n^{-1}(\tilde{\underset{\sim}{b}}) - D_n^{-1}(\underset{\sim}{b})| \leq n^{-h}, \quad n \geq n_0 \tag{21}$$

for some n_0 not depending on $\underset{\sim}{b}$, $\tilde{\underset{\sim}{b}}$. Where for any vector or matrix C, $|C|$ denotes the maximum module of the elements of C.

Proof. Since

$$D_n^{-1}(\tilde{\underset{\sim}{b}}) - D_n^{-1}(\underset{\sim}{b}) = D_n^{-1}(\tilde{\underset{\sim}{b}})\left(D_n(\underset{\sim}{b}) - D_n(\tilde{\underset{\sim}{b}})\right)D_n^{-1}(\underset{\sim}{b}),$$

(21) follows easily from Lemma 1.

In the following we use $\chi_n^2(\delta)$ to denote the noncentral Chi-square distribution with degree of freedom n and noncentrality parameter δ. $\chi_n^2(0)$ will be abbreviated to χ_n^2.

LEMMA 3. Suppose that $\{\xi_n\}$ is a sequence of random variables, ξ_n is distributed as $\chi_n^2(\delta_n)$, and there exists positive constants $\eta_1 \le \eta_2$ such that

$$\eta_1 n \le \delta_n^2 \le \eta_2 n, \quad n = 1,2,\ldots .$$

Then we can find positive constant c independent of n, such that

$$P(\xi_n/n > 1 + \eta_1/2) \ge 1 - e^{-cn}, \quad n = 1,2,\ldots . \tag{22}$$

Proof. We can find random variables $\xi \sim \chi_n^2$, $Z \sim N(0,1)$, such that ξ_n is distributed as $\xi + 2\delta_n Z + \delta_n^2$. Choose $\varepsilon \in (0, \eta_1\eta_2^{-1/2}/8)$, we have

$$P(|Z| \ge \varepsilon\sqrt{n}) \le \frac{2}{\sqrt{2\pi}\,\varepsilon\sqrt{n}} \exp(-\frac{\varepsilon^2 n}{2}) \le e^{-c_1 n}$$

for $c_1 = \varepsilon^2/2$, $n \ge 2/(\pi\varepsilon^2)$. But when $|Z| < \varepsilon\sqrt{n}$, we have

$$|2\delta_n Z| \le 2\sqrt{\eta_2}\,\varepsilon n \le \eta_1 n/4.$$

Therefore,

$$P(\xi_n/n > 1 + \eta_1/2) \ge P(\xi/n > 1 + \eta_1/4) - P(|Z| \ge \varepsilon\sqrt{n})$$

$$\ge 1 - P(|\xi/n - 1| \ge \eta_1/4) - e^{-c_1 n}, \quad n \ge 2/\varepsilon^2. \tag{23}$$

Since ξ is the sum of iid. variables $X_1^2, \ldots, X_n^2$, with $X_1 \sim N(0,1)$, in view of the fact that X_1^2 has moment generating function in some neighborhood of zero, it is well-known ([3], p.288) that there exists a constant $c_2 > 0$ such that

$$P(|\xi/n - 1| \ge \eta_1/4) \le e^{-c_2 n}, \quad \text{for} \quad n \ge n_1.$$

From this and (23), we see that (22) holds for $c = \min(c_1, c_2)$, when $n \ge \max(2/\varepsilon^2, n_1)$. Replacing c by some smaller quantity, we can make (22) true for all n.

3. PROOF OF THE THEOREM

Introduce the following notations:

$$\underset{\sim}{s}_j = (s_{1j}, \ldots, s_{pj})', \qquad w_k(\underset{\sim}{b}) = b_0 + b_1\lambda_k + \ldots + b_p\lambda_k^p,$$

$$\Lambda_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_p \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \lambda^{n-1} & \lambda^{n-1} & \cdots & \lambda_p^{n-1} \end{pmatrix},$$

$$\delta^2(n,\underset{\sim}{b}) = \sum_{j=1}^{N} \underset{\sim}{s}_j^* \Lambda_n^* B_n^*(\underset{\sim}{b}) D_n^{-1}(\underset{\sim}{b}) B_n(\underset{\sim}{b}) \Lambda_n \underset{\sim}{s}_j .$$

For simplicity of writing and without losing generality, suppose that $\sigma^2 = 1$. Then from the assumptions imposed on $\{e_j(t)\}$, we have

$$Q_n(Y,b) \sim \chi^2_{2Nm}\left(\delta^2(n,\underset{\sim}{b})\right).$$

Remember that $m = n - p$. From (9) and the fact that $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_p| = 1$, we have

$$\delta^2(n,\underset{\sim}{b}) \geq \frac{1}{p+1} \sum_{j=1}^{N} |B_n(\underset{\sim}{b})\Lambda_n \underset{\sim}{s}_j|^2 = \frac{1}{p+1} \sum_{j=1}^{N} \sum_{t=0}^{m-1} \left|\sum_{k=0}^{p} \lambda_k^t w_k(\underset{\sim}{b}) s_{kj}\right|^2$$

$$= \frac{1}{p+1}\left\{ \sum_{j=1}^{N} \sum_{t=0}^{m-1} \sum_{k=1}^{p} |w_k(\underset{\sim}{b})|^2 \, |s_{kj}|^2 \right.$$

$$\left. + \sum_{u\neq v}^{p} \sum_{j=1}^{N} \sum_{t=0}^{m-1} (\lambda_u \bar{\lambda}_v)^t w_u(\underset{\sim}{b}) \bar{w}_v(\underset{\sim}{b}) s_{uj} \bar{s}_{vj} \right\}. \qquad (24)$$

By assumption

$$\alpha \equiv \min\{\sum_{j=1}^{N} |s_{kj}|^2 : k = 1,\ldots,p\} > 0.$$

Hence,

$$\sum_{j=1}^{N} \sum_{t=0}^{m-1} \sum_{k=1}^{p} |w_k(\underset{\sim}{b})|^2 |s_{kj}|^2 \geq \alpha(n-p) \sum_{k=1}^{p} |w_k(\underset{\sim}{b})|^2. \tag{25}$$

Put

$$s = \max\{|s_{kj}|: 1 \leq k \leq p,\ 1 \leq j \leq N\},$$

$$\lambda = \min\{|\lambda_i - \lambda_j|: 1 \leq i < j \leq p\}.$$

We have $\lambda > 0$ since by assumption $\lambda_i \neq \lambda_j$ when $i \neq j$. Thus

$$\left|\sum_{t=0}^{m-1} (\lambda_u\ \overline{\lambda}_v)^t\right| \leq \frac{2}{1-\lambda}, \quad n \geq p+1, \quad u \neq v.$$

Therefore, noticing that $|w_k(\underset{\sim}{b})| \leq \sqrt{p+1}$, we have

$$\left|\sum_{u\neq v}^{p} \sum_{j=1}^{N} \sum_{t=0}^{m-1} (\lambda_u \overline{\lambda}_v)^t w_u(\underset{\sim}{b}) \overline{w}_v(\underset{\sim}{b}) s_{uj} \overline{s}_{vj}\right| \leq 2p^2(p+1)Ns^2(1-\lambda)^{-1}. \tag{26}$$

Define the set

$$B_\varepsilon = \left\{\underset{\sim}{b}: \underset{\sim}{b} = (b_0, \ldots, b_p)' \in B,\ \sum_{k=1}^{p} |b_k - b_k^{(0)}|^2 \geq \varepsilon\right\}.$$

Since $\{\lambda_1, \ldots, \lambda_p\}$ is the set of all roots of $b_0^{(0)} + b_1^{(0)}Z + \ldots + b_p^{(0)}Z^p = 0$, it is easily seen that

$$\inf\left\{\sum_{k=1}^{p} |w_k(\underset{\sim}{b})|^2 : \underset{\sim}{b} \in B_\varepsilon\right\} > 0. \tag{27}$$

Summing up (24)-(27), we see that there exists constants $\eta_1 > 0$, $\eta_2 > 0$ depending only on ε, such that

$$\eta_1 n \leq \delta^2(n,\underset{\sim}{b}) \leq \eta_2 n, \quad \underset{\sim}{b} \in B_\varepsilon. \tag{28}$$

To simplify the wording, in the sequel the symbol c will be used to denote any positive constant not depending on $\underset{\sim}{b}$, n, which may assume different values on each of its appearances, and the phrase "for n large" means that "for n larger than some n_0 independent of $\underset{\sim}{b} \in B$". Since $Q_n(Y,\underset{\sim}{b}) \sim \chi^2_{2Nm}\left(\delta^2(n,\underset{\sim}{b})\right)$, from (28) and Lemma 3, we obtain

$$P\left(Q_n(Y,\underset{\sim}{b})\big/nN \geq 1 + \eta_1/2\right) \geq 1 - e^{-cn}. \tag{29}$$

Choose $h_1 > 0$ according to some $h > 0$ as in Lemma 2. The value of h will be specified later. Choose a subset $B_{\varepsilon n}$ of B_ε with no more than n^{2ph_1} points, such that for each $\underset{\sim}{b} \in B_\varepsilon$, there exists $\tilde{\underset{\sim}{b}} \in B_{\varepsilon n}$ such that $|\tilde{\underset{\sim}{b}} - \underset{\sim}{b}| \leq n^{-h_1}$. From (44), for n large, we have

$$P\left(\min_{\underset{\sim}{b} \in B_{\varepsilon n}} Q_n(Y,\underset{\sim}{b})/nN \geq 1 + \eta_1/2\right) \geq 1 - n^{2ph_1}e^{-cn} \geq 1 - e^{-cn}. \tag{30}$$

Now choose arbitrarily $\underset{\sim}{b} \in B_\varepsilon$. Find $\tilde{\underset{\sim}{b}} \in B_{\varepsilon n}$ such that $|\tilde{\underset{\sim}{b}} - \underset{\sim}{b}| \leq n^{-h_1}$. Consider

$$J = |Q_n(Y,\underset{\sim}{b}) - Q_n(Y,\tilde{\underset{\sim}{b}})|. \tag{31}$$

Abbreviating $B_n(\underset{\sim}{b})$, $B_n(\tilde{\underset{\sim}{b}})$, etc. to B_n, $\tilde{B}_n$, etc., we have

$$\begin{aligned} J &\leq \sum_{j=1}^{N} \left|\underset{\sim}{Y}^*(j,n)B_n^*D_n^{-1}B_n\underset{\sim}{Y}(j,n) - \underset{\sim}{Y}^*(j,n)\tilde{B}_n^*\tilde{D}_n^{-1}\tilde{B}_n\underset{\sim}{Y}(j,n)\right| \\ &\leq \sum_{j=1}^{N} \left|\underset{\sim}{Y}^*(j,n)B_n^*(\tilde{D}_n^{-1} - D_n^{-1})B_n\underset{\sim}{Y}(j,n)\right| \\ &\quad + \sum_{j=1}^{N} \left|\underset{\sim}{Y}^*(j,n)\tilde{B}_n^*\tilde{D}_n^{-1}\tilde{B}_n\underset{\sim}{Y}(j,n) - \underset{\sim}{Y}^*(j,n)B_n^*\tilde{D}_n^{-1}B_n\underset{\sim}{Y}(j,n)\right| \\ &\equiv J_1 + J_2. \end{aligned} \tag{32}$$

By (21), we have

$$J_1 \le n^{-h} \sum_{j=1}^{N} \|B_n \underset{\sim}{Y}(j,n)\|^2 \le n^{-h} \sum_{j=1}^{N} \sum_{t=0}^{n-1} (p+1)|Y_j(t)|^2.$$

Put $\tilde{s} = \max\{\sum_{i=1}^{p}|s_{ij}|: j = 1,\ldots,N\}$. Since $|\lambda_1| = \ldots = |\lambda_p| = 1$, we have

$$\left|\sum_{i=1}^{p} s_{ij}\lambda_i^t\right| \le \tilde{s}, \quad j = 1,\ldots,N.$$

Hence,

$$J_1 \le n^{-h}\left\{2nN(p+1)\tilde{s}^2 + 2\sum_{j=1}^{N}\sum_{t=0}^{n-1}(p+1)|e_j(t)|^2\right\}$$

$$\le Cn^{-h+1} + Cn^{-h}\sum_{j=1}^{N}\sum_{t=0}^{n-1}|e_j(t)|^2. \tag{33}$$

Introduce the event

$$E_n = \left\{\sum_{j=1}^{N}\sum_{t=0}^{n-1}|e_j(t)|^2 \le 2nN\right\}.$$

By (33), we have

$$E_n \subset \left\{J_1 \le Cn^{-h+1}\right\}. \tag{34}$$

For J_2, we have

$$J_2 \le \sum_{j=1}^{N}\left|\underset{\sim}{Y}^*(j,n)(\tilde{B}_n^* - B_n^*)\tilde{D}_n^{-1}\tilde{B}_n\underset{\sim}{Y}(j,n)\right|$$

$$+ \sum_{j=1}^{N}\left|\underset{\sim}{Y}^*(j,n)B_n^*\tilde{D}_n^{-1}(\tilde{B}_n - B_n)\underset{\sim}{Y}(j,n)\right| \equiv J_3 + J_4. \tag{35}$$

By the extended Schwarz inequality,

$$J_3^2 \le N \sum_{j=1}^{N} \underset{\sim}{Y}^*(j,n)\tilde{B}_n^* D_n^{-1} B_n \underset{\sim}{Y}(j,n) \underset{\sim}{Y}^*(j,n)(\tilde{B}_n^* - B_n)\tilde{D}_n^{-1}(\tilde{B}_n - B_n)\underset{\sim}{Y}(j,n). \quad (36)$$

Write $w = 3(p+1)^2 + 1$. By (10), (36), we have for n large,

$$J_3^2 \le N \sum_{j=1}^{N} n^{2w} \| B_n \underset{\sim}{Y}(j,n) \|^2 \| (\tilde{B}_n - B_n)\underset{\sim}{Y}(j,n) \|^2$$

$$\le N n^{2w} \sum_{j=1}^{N} \| B_n \underset{\sim}{Y}(j,n) \|^2 \sum_{j=1}^{N} \| (\tilde{B}_n - B_n)\underset{\sim}{Y}(j,n) \|^2. \quad (37)$$

In the course of proving (33), we have shown that

$$\sum_{j=1}^{N} \| B_n \underset{\sim}{Y}(j,n) \|^2 \le Cn + C \sum_{j=1}^{N} \sum_{t=0}^{n-1} |e_j(t)|^2. \quad (38)$$

Further, in view of $|\tilde{B}_n - B_n| \le n^{-h_1}$, we have

$$\sum_{j=1}^{N} \| (\tilde{B}_n - B_n)\underset{\sim}{Y}(j,n) \|^2 \le n^{-2h_1}(p+1)^2 \sum_{j=1}^{N} \sum_{t=0}^{n-1} |Y_j(t)|^2$$

$$\le Cn^{-2h_1+1} + Cn^{-2h_1} \sum_{j=1}^{N} \sum_{t=0}^{n-1} |e_j(t)|^2. \quad (39)$$

From (37)-(39), we see that

$$E_n \subset \left\{ J_3 \le Cn^{-(h_1-w-1)} \right\}. \quad (40)$$

Likewise, we obtain

$$E_n \subset \left\{ J_4 \le Cn^{-(h_1-w-1)} \right\}. \quad (41)$$

Summing up (31), (33), (35), (40) and (41), we obtain

$$E_n \subset \{ |Q_n(Y,\underset{\sim}{b}) - Q_n(Y,\tilde{\underset{\sim}{b}})| \le Cn^{-h+1} + Cn^{-h_1+w+1} \}. \quad (42)$$

Now we choose $h = 1$. Choose h_1 corresponding to this h according to Lemma 2 such that $h_1 \geq w + 1$. For this choice of h and h_1, from (30) and (42), we get for n large

$$P\left(\min_{\underset{\sim}{b} \in B_\varepsilon} Q_n(Y,\underset{\sim}{b})/(nN) \geq 1 + \eta_1/2\right) \geq 1 - e^{-cn} - \left(1 - P(E_n)\right). \tag{43}$$

On the other hand, since $Q_n(Y,\underset{\sim}{b}^{(0)}) \sim \chi^2_{2nN}$, we have for n large,

$$P\left(Q_n(Y,\underset{\sim}{b}^{(0)}) \leq 1 + \eta_1/4\right) \geq 1 - e^{-cn}. \tag{44}$$

Likewise, since $\sum_{j=1}^{N}\sum_{t=0}^{n-1}|e_j(t)|^2 \sim \chi^2_{2nN}$, we have for n large,

$$P(E_n) \geq 1 - e^{-cn}. \tag{45}$$

Summing up (43)-(45), we obtain for n large,

$$P\left(\min_{\underset{\sim}{b} \in B_\varepsilon} Q_n(Y,\underset{\sim}{b}) > Q_n(Y,\underset{\sim}{b}^{(0)})\right) \leq e^{-cn}.$$

which entails (7), and the theorem is proved.

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